On Non-harmonic Trigonometrical Series.

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§ 1. In preceding communications I have indicated the important consequences for the theory of series of functions of the consideration of what I have called Restricted Fourier series. This is due to the fact that a large proportion of the series actually employed in analysis can be replaced, to a first approximation, by trigonometrical series of the harmonic type. This is the case, for example, with series of Legendre coefficients, and with series of Bessel functions of rational order.

There are, however, classes of series which can only be approximated to by non-harmonic trigonometrical series. This is, for example, the case with series of Bessel functions of irrational order. The typical term of the trigonometrical series is no longer $\cos n(x+\alpha)$, where n is an integer, but $\cos k_n(x+\alpha)$, where, however, successive values of k_n tend to differ by a constant quantity, which may, of course, without loss of generality, be taken to be unity.*

We are thus led to attempt to apply to series of the type

$$\sum_{n=1}^{\infty} A_n^{(k)} \equiv \sum_{n=1}^{\infty} \{a_n \cos(k+n)x + b_n \sin(k+n)x\},$$

where k is irrational, the considerations exposed in our discussion of Restricted Fourier series.

The object of the present communication is to show in what manner this step in what may be called the new theory of the convergence of series of functions may be taken. The fundamental theorem obtained is the following :—

THEOREM.—*If the trigonometrical series*

$$\tfrac{1}{2}A_0^{(k)} + \sum_{n=1}^{\infty} A_n^{(k)} = \tfrac{1}{2}a_0 \cos kx + \sum_{n=1}^{\infty} \{a_n \cos(k+n)x + b_n \sin(k+n)x\}, \quad (1)$$

is such that (i) *its second integrated series converges uniformly to* $G(x)$ *and* (ii) *at the point* x *the individual term* $A_n^{(k)}$ *tends to zero,* $(n \to \infty)$, *then the*

* The conclusions thus obtained refer primarily to the internal points of the intervals associated with the series. There is, however, usually no difficulty in extending them to such end-points as are not singular points for which the asymptotic trigonometric approximation fails. Thus, in the case of Bessel series, it is only the origin which is to be excepted ; at the point $x = 1$ the series is subject to the same laws as at points in the interior of (0, 1).

upper and lower functions of the original series at the point x are independent of the behaviour of the series outside an arbitrary small neighbourhood surrounding the point x considered.

Moreover, if the point x is internal to a completely open interval in which $G(x)$ is the integral of an integral, the conditions for convergence of the series (1) *at the point x are the same as those of the Fourier series of a function equal to $f(x) = d^2G/dx^2$ almost everywhere in the neighbourhood in question, and otherwise arbitrary.*

I do not, in the present note, enter into the question of expanding a function in a series $\Sigma A_n^{(k)}$, but must refer my readers to a communication on this subject to be made elsewhere. Yet I may say that, in this respect also, the series in question behave, in the interior of an interval of length 2π, in the same manner as harmonic trigonometrical series, *provided n is allowed to assume all negative, as well as all positive, integral values.* For generality this latter assumption is made in the present paper, though in the applications we have here in view it is sufficient to suppose n to be positive and zero only.*

§ 2. When k is rational, a suitable change of the variable reduces the series $\Sigma A_n^{(k)}$ to a harmonic series. There are, therefore, essentially only two cases to be discussed, that in which k is zero, and that in which k is irrational. The former case I have already considered at length in a communication to the London Mathematical Society.† For the convenience of my readers I propose in the present note to indicate briefly the nature of the argument employed, and at the same time to slightly extend the scope in the sense of the general enunciation of the preceding article. This seems desirable in view of the fact that the reasoning given below for the irrational case does not apply as it stands when k is zero.

AUXILIARY THEOREM.—*If*

$$G(x) \sim -\sum_{n=1}^{\infty} n^{-2}A_n = -\sum_{n=1}^{\infty} n^{-2}(a_n \cos nx + b_n \sin nx)$$

and $\phi(u)$ is the integral of an integral of an integral in the closed interval $(0, \pi)$ and is zero at the end-points of this interval, then

$$\frac{d^2}{dx^2}\int_0^{\pi} \{G(x+u) + G(x-u)\}\,\phi(u) \sin nu\, du \to 0, \qquad (n \to \infty) \qquad (I)$$

provided only $A_n \to 0, \qquad (n \to \infty).$

* The omission of the negative indices in the present paper would only impair the theoretical generality of the result. In the general theory of the series their omission leads of course to a serious restriction on the nature of the function which can be represented by such a series. In fact, the elegance and simplicity of the theory and the analogy with Fourier series all disappear. It is worth noting, however, that the cosine and sine series require to be considered separately.

† "On Restricted Fourier Series and the Convergence of Power Series," 1917.

For $\phi(u)$ may then be expanded in the form

$$\phi(u) = \sum_{p=1}^{\infty} c_p \sin pu,$$

in the closed interval $(0, \pi)$. This series will then converge absolutely and uniformly, and the first derived series will converge uniformly to $\phi'(u)$, while the second derived series converges boundedly, the sum-function being in $(0 < u < \pi)$ equal to $\phi''(u)$. These sum-functions have therefore bounded variation in $(0, \pi)$. We may, accordingly, in the following calculations, first perform the differentiation under the integral sign and then substitute these series term-by-term for their sum-functions.

We have

$$\tfrac{1}{2}\{G(x+u)+G(x-u)\} \sim -\sum_{n=1}^{\infty} n^{-2} A_n \cos nu,$$

and therefore, writing $H(x) = \int dx \int G(x)\, dx$, and using twice integration by parts,

$$\begin{aligned}
&\frac{d^2}{dx^2}\int_0^{\pi}\{G(x+u)+G(x-u)\}\phi(u)\sin nu\, du \\
&\quad = -\frac{d^2}{dx^2}\int_0^{\pi}\{H(x+u)+H(x-u)\}\frac{d^2}{du^2}\{\phi(u)\sin nu\}\, du \\
&\quad = -\sum_{p=1}^{\infty} c_p \int_0^{\pi}\{G(x+u)+G(x-u)\}\frac{d^2}{du^2}(\sin pu \sin nu)\, du \\
&\quad = \sum_{p=1}^{\infty} c_p \frac{d^2}{dx^2}\int_0^{\pi}\{G(x+u)+G(x-u)\}\tfrac{1}{2}\{\cos(p-n)u - \cos(p+n)u\}\, du \\
&\quad = \sum_{p=1}^{n-1} c_p \tfrac{1}{2}\pi A_{n-p} + \sum_{p=n}^{\infty} c_p \tfrac{1}{2}\pi A_{p-n} - \sum_{p=1}^{\infty} c_p \tfrac{1}{2}\pi A_{n+p},
\end{aligned}$$

which is easily shown to tend to zero, since $\sum_{p=1}^{\infty} |c_p|$ is convergent, and $A_n \to 0$, $(n \to \infty)$.

This proves the theorem.

COR.—*If in place of the condition* $A_n \to 0$, *we have the pair of conditions*

$$a_n \to 0, \qquad b_n \to 0, \qquad (n \to \infty),$$

the convergence of our expression (I) *to its limit zero is uniform.*

In this case, moreover, integration by parts being no longer needed, it is sufficient to assume that $\phi(u)$ is the integral of an integral, vanishing at the origin.

§ 3. Using this auxiliary theorem we at once have the following:—

THEOREM.—*The upper and lower functions at the point* x *of the second derived series of the Fourier series of a function* $G(x)$, *depend only on the nature of* $G(x)$

in an arbitrary small neighbourhood enclosing the point x considered, provided merely the individual term $A_n = a_n \cos nx + b_n \sin nx$, of the second derived series in question, tend to zero as $n \to \infty$.

In fact, summing the Fourier series in the usual way, we get for the partial summation of our series the following expressions:—

$$\frac{1}{2\pi}\frac{d^2}{dx^2}\int_0^\pi \{G(x+u)+G(x-u)\}\sin(n+\tfrac{1}{2})u \operatorname{cosec}\tfrac{1}{2}u\,du$$

$$= \tfrac{1}{2}A_n + \frac{1}{2\pi}\frac{d^2}{dx^2}\int_0^\pi \{G(x+u)+G(x-u)\}\cot\tfrac{1}{2}u \sin nu\,du.$$

Using the auxiliary theorem, and giving to $\phi(u)$ the expression $\cot\frac{1}{2}u$ in the interval (e, π), the required result at once follows, the upper and lower functions being thus seen to be the upper and lower limits, as $n \quad \infty$, of

$$\frac{1}{2\pi}\frac{d^2}{dx^2}\int_0^e \{G(x+u)+G(x-u)\}\{\cot\tfrac{1}{2}u-\phi(u)\}\sin nu\,du$$

where $\phi(u)$ is an arbitrary function, which is the integral of an integral of an integral in $(0, e)$, and has the value 0 at the origin and $\cot\frac{1}{2}e$ at the point e.

§ 4. Using the corollary of § 2 we have a corresponding theorem, in which, since the first derived series is now a Fourier series, we may perform one of the differentiations under the integral sign, and write for the expression whose limits have to be considered

$$\frac{d}{dx}\int_0^e \{F(x+u)+F(x-u)\}\{\cot\tfrac{1}{2}u-\phi(u)\}\sin nu\,du,$$

where $G(x) = \int F(x)\,dx$.

§ 5. If in addition $G(x)$ is the integral of an integral in some interval, however small, surrounding the point x considered, we may perform the remaining differentiation under the integral sign. By the Theorem of Riemann-Lebesgue, the part of our expression involving the arbitrary function vanishes when $n \to \infty$, and we are left with an expression formally the same as for a Fourier series. The series itself has at the same time become an ordinary restricted Fourier series or R. F. series.

We have thus obtained as a corollary to the theorem of § 4, the result already given by me in the paper quoted.

§ 6. If in the auxiliary theorem of § 2 we take the interval $(-\pi, \pi)$ in place of $(0, \pi)$, we may take either $\sin nu$ or $\cos nu$ as a factor of the integrand, since, $G(x+u) + G(x-u)$ being an even function, the new terms introduced are all zero. Moreover, if $a_n \to 0$, $b_n \to 0$, we need not demand more of $\phi(u)$ than that it should be the integral of an integral in $(-\pi, \pi)$, the values at 0, π, and $-\pi$ being unrestricted.

Similarly we can take for interval of an integration any interval $[2(r-1)\pi,\ 2(s+1)\pi]$, since the integral expressing A_n may equally well be taken to have these limits, provided we divide by an irrelevant numerical factor.

Now suppose $\phi(u)$ to be zero, except in the open interval (b, c) where $2(r-1)\pi < b < c < 2(s+1)\pi$. Then since $\phi(u)$ is the integral of an integral in $[2(r-1)\pi, 2(s+1)\pi]$, $\phi'(u)$ is also zero, except in the same open interval. Thus *in the corollary to our auxiliary theorem* (§ 2), *we may take the interval of integration to be* (b, c) *in place of* $(0, \pi)$, *provided in* (b, c) ϕ *is the integral of an integral, and*

$$\phi(b) = \phi(c) = \phi'(b) = \phi'(c) = 0,$$

moreover, in (I) *we may write* $\cos n(u-\alpha)$ *in place of* $\sin nu$.

Also, if $a_n \to 0$, $b_n \to 0$, the factor $G(x+u) + G(x-u)$ may evidently be replaced by $G(x+u) - G(x-u)$, and therefore we may write

$$\frac{d^2}{dx^2}\int_b^c G(x \pm u) + \phi(u) \cos n(u-\alpha)\, du \to 0.$$

§ 7. In the auxiliary theorem and corollary, as well as in § 6, we may replace the differentiation sign d^2/dx^2 by the factor n^2, in virtue of the following calculation, *provided* G *is an integral and* ϕ *and* ϕ' *are zero at the limits of integration* :—

$$\frac{d^2}{dx^2}\int_b^c G(x+u)\,\phi(u)\cos\mu(u-\alpha)\,du$$

$$= \frac{d^2}{dx^2}\int_{b+x}^{c+x} G(v)\,\phi(v-x)\cos\mu(v-x-\alpha)\,dv$$

$$= \int_{b+x}^{c+x} G(v)\,\frac{d^2}{dx^2}\{\phi(v-x)\cos\mu(v-x-\alpha)\}\,dv,$$

since, when we differentiate once, the part due to the limits of integration is zero, as $\phi(b) = \phi(c) = 0$, and when we differentiate a second time the part due to the limits of integration is zero, since $\phi'(b) = \phi'(c) = 0$. Thus our expression becomes

$$\int_b^c G(x+u)\,\phi''(u)\cos\mu(u-\alpha)\,du - 2\int_b^c G(x+u)\,\phi'(u)\,\mu\sin\mu(u-\alpha)\,du$$

$$-\mu^2\int_b^c G(x+u)\,\phi(u)\cos\mu(u-\alpha)\,du.$$

The first of these integrals vanishes, when $\mu \to \infty$, by the Theorem of Riemann-Lebesgue. The second, when integrated by parts, becomes

$$\int_b^c \cos\mu(u-\alpha)\,\frac{d}{du}\{G(x+u)\,\phi'(u)\}\,du,$$

which vanishes, when $\mu \to \infty$, by the Theorem of Riemann-Lebesgue, since G and ϕ' are both integrals, and therefore $\frac{d}{du}\{G(x+u)\phi'(u)\}$ is summable.

The same form of argument applies when $G(x+u)$ is replaced by $G(x-u)$.

Thus

$$\operatorname*{Lt}_{\mu\to\infty} \frac{d^2}{dx^2}\int_b^c G(x\pm u)\,\phi(u)\cos\mu(u-\alpha)\,du$$
$$= -\operatorname*{Lt}_{n\to\infty} \mu^2 \int_b^c G(x\pm u)\,\phi(u)\cos\mu(u-\alpha)\,du.$$

This proves the required result.

§ 8. In the form given to our theorem in § 7, the student of Riemann will recognise the close connection of our auxiliary theorem with what Riemann calls his Third Theorem on Trigonometrical Series.* Clearing the enunciation of restrictions which are now seen to have been unnecessary,† the theorem is as follows :—

RIEMANN'S THIRD THEOREM.—*If in* (b, c) *the function* $\lambda(x)$ *is the integral of an integral, and if* $\lambda(x)$ *and* $\lambda'(x)$ *are both zero at* b *and* c, *then the integral*

$$\mu^2 \int_b^c G(x)\cos\mu(x-\alpha)\,\lambda(x)\,dx \to 0, \qquad (\mu \to \infty),$$

where

$$G(x) = -\sum_{n=1}^{\infty} n^{-2}(a_n \cos nx + b_n \sin nx) = -\sum_{n=1}^{\infty} n^{-2} A_n,$$

and the coefficients satisfy the conditions

$$a_n \to 0, \qquad b_n \to 0.$$

Here μ is not necessarily, as it was in our auxiliary theorem, an integer.

By the argument of § 7 we may, in Riemann's Theorem, replace the factor μ^2 by the sign of differentiation d^2/dx^2.

§ 9. We now proceed to indicate how the extended form of Riemann's Third Theorem, applicable to our non-harmonic series, may be obtained. We first remark that Riemann has proved that the series

$$\frac{1}{\mu}\Sigma \frac{1/\mu}{(n/\mu)^2(1-n/\mu)^2},$$

where n assumes all positive and negative integral numbers except a finite (fixed) number in the neighbourhood of zero, and in the neighbourhood of μ, determined by the inequalities

$$-c' < n < c'', \qquad \mu - c''' < n < \mu + c'''',$$

* B. Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," 'Ges. Werke,' pp. 248–251 (1854).

† It is not even necessary that $\lambda''(x)$ should be bounded.

has a sum which, as $\mu \to \infty$, remains bounded. In fact, if we divide the whole infinite straight line up into parts of length $1/\mu$ right and left from the origin and form the *lower* summation with respect to these intervals of the function $x^{-2}(1-x)^{-2}$, from $-\infty$ to $+\infty$, omitting the intervals

$$\left(-\frac{c'-1}{\mu} < x < \frac{c''-1}{\mu}\right) \quad \text{and} \quad \left(1-\frac{c'''-1}{\mu}\right) < x < \left(1-\frac{c''''-1}{\mu}\right),$$

the c's being supposed each > 1, we get a summation which *contains* every term of our series $\Sigma \frac{1/\mu}{(n/\mu)(1-n/\mu)^2}$ *and is not greater than*

$$\int \frac{dx}{(1-x)^2 x^2} = \left[-\frac{1}{x} + \frac{1}{1-x} + 2\log\frac{x}{1-x}\right]$$

over the same intervals.

Putting in the values corresponding to the limits of integration and dividing by μ, we easily see that we get a quantity which remains bounded as $\mu \to \infty$.

This proves the required result, which may be expressed by saying that

$$\Sigma \frac{\mu^2}{n^2(\mu-n)^2}$$

remains bounded as $\mu \to \infty$, *the summation being made for all positive and negative values of* n, *omitting a finite number on each side of the values* 0 *and* μ.

§ 10. The enunciation of the theorem is as follows:—

THEOREM.—*If in* (b, c) *the function* $\lambda(x)$ *is the integral of an integral, and if* $\lambda(x)$ *and* $\lambda'(x)$ *are both zero at* b *and* c, *then*

$$\mathrm{I} \equiv \mu^2 \int_b^c \{\mathrm{G}(x+u) + \mathrm{G}(x-u)\}\, \lambda(u) \begin{matrix}\cos \\ \sin\end{matrix} \mu u\, du \to 0, \qquad (\mu \to \infty),$$

where

$$\mathrm{G}(x) = -\sum_{n=-\infty}^{n=+\infty} (n+k)^{-2}\{a_n \cos(n+k)x + b_n \sin(n+k)x\}$$

$$= -\sum_{n=-\infty}^{n=+\infty} (n+k)^{-2} \mathrm{A}_n^{(k)},$$

where $0 \quad k < 1$, *provided*

$$\mathrm{A}_n^{(k)} \to 0, \qquad (n \to \infty),$$

and the series converges uniformly.

We shall prove the theorem for $\cos \mu u$, the proof for $\sin \mu u$ is similar. Since the series converges uniformly, we may replace $\{\mathrm{G}(x+u) + \mathrm{G}(x-u)\}$ in I by its equivalent series. We thus get I expressed by means of a series (Φ), whose generic term is

$$-\frac{\mu^2}{(n+k)^2} \mathrm{A}_n^{(k)} \int_b^c \cos(n+k)\, u \cos \mu u \;.\; \lambda(u)\, du.$$

If $(\mu-k)$ is not an integer, or, being an integer, is not n, we may write this

$$-\tfrac{1}{2}\frac{\mu^2}{(n+k^2)}\,A_n\int_b^c\{\cos(n+k+\mu)u+\cos(n+k-\mu)u\}\,\lambda(u)\,du$$

$$=\tfrac{1}{2}\frac{\mu^2}{(n+k)^2(\mu+n+k)^2}\,A_n\int_b^c\cos(n+k+\mu)u\,.\,\lambda''(u)\,du$$

$$+\tfrac{1}{2}\frac{\mu^2}{(n+k)(\mu-n-k)^2}\,A_n\int_b^c\cos(\mu-n-k)u\,.\,\lambda''(u)\,du,$$

where we have integrated by parts twice, and used the condition

$$\lambda(b)=\lambda(c)=\lambda'(b)=\lambda'(c)=0.$$

Also if $\left|\,\mu\pm n\,\right|\geq 1$,

$$(\mu+n+k)^2\geq(\mu+n)^2(1-k)^2,\qquad(\mu-n-k)^2\geq(\mu-n)^2(1-k)^2.$$

Thus we see that the generic term of (Φ) is numerically

$$\leq U_\mu(1-k)^{-2}\left\{\frac{\mu^2}{n^2(\mu+n)^2}+\frac{\mu^2}{n^2(\mu-n)^2}\right\},$$

where U_μ denotes, for fixed μ, the *greatest* of the quantities

$$A_n\int_b^c\cos\{\mu\pm(k+n)\}u\,.\,\lambda''(u)\,du$$

n assuming all positive integral values such that

$$\left|\,\mu\pm n\,\right|\geq 1.$$

Such a *greatest* quantity exists, since, by the theorem of Riemann-Lebesgue, these quantities tend to zero as $n\to\infty$. By the same theorem, n varying with μ, as it does in U_μ, the integral in the expression tends to zero, when $\mu\to\infty$, however n may vary, unless $(\mu-n)$ remains finite; while, if $(\mu-n)$ remains finite, $n\to\infty$ and therefore $A_n\to 0$. Thus, however n varies in the expression for U_μ, we have

$$U_\mu\to 0,\qquad\mu\to\infty.$$

Also, the individual terms of (Φ) tend to zero as $\mu\to\infty$ by the same reasoning, therefore the omission of any finite number of them, fixed as μ varies, will not affect our result. Let us omit a finite number greater than unity on each side of the origin and the quantity $\pm\mu$, the value $\pm\mu-k$ will then, if an integer, be eliminated, and we shall certainly have $|\mu\pm n|>1$. What remains of (Φ) will then, by what has been shown, be numerically

$$\leq 2U_\mu(1-k)^{-e}\sum\frac{\mu^2}{n^2(\mu-n)^2}$$

the summation being made for all positive and negative integers n, except a finite number on each side of the values of 0 and $\pm\mu$.

As pointed out in § 9, this summation is bounded as $\mu \to \infty$, therefore, since $U_\mu \to 0$, our expression tends to zero as $\mu \to \infty$. This proves the theorem.

COR.—*If, in addition to the conditions of the theorem,* $G(x)$ *is an integral,*

$$\frac{d^2}{dx^2} \cdot \int_b^c \{G(x+u)+G(x-u)\}\lambda(u)\cos\mu(u-\alpha)\,du \to 0, \qquad (\mu \to \infty).$$

This follows at once by § 7.

Here one of the differentiations may be performed under the integral sign.

§ 11. Similarly, if, instead of $A_n^{(k)} \to 0$, we have $B_n^{(k)} \to 0$, where

$$B_n^{(k)} = b_n \cos(n+k)x - a_n \sin(n+k)x,$$

we have corresponding results with $G(x+u)-G(x-u)$ in place of $G(x+u)+G(x-u)$.

§ 12. If $a_n \to 0$, $b_n \to 0$, the results of §§ 10 and 11 both hold, thus,

$$\mu^2 \int_b^c G(x \pm u)\lambda(u)\cos\mu(u-\alpha)\,du \to 0, \qquad (\mu \to \infty),$$

which, when we suppress the x, is the form adopted by Riemann, and constitutes the exact generalisation of his Third Theorem, and coincides with it when we put $k = 0$.

In this case, when $a_n \to 0$, $b_n \to 0$, the condition as to uniform convergence is satisfied of itself. Moreover, G *is now an integral,* for we may write its expansion in the form

$$G(x) - k^{-2}A_0^{(k)} = \cos kx\,(T_1+T_2) + \sin kx\,(T_3+T_4),$$

where, for instance,

$$T_1 = \sum_{r=1}^{\infty} r^{-2}(1+k/r)^{-2}A_r^{(k)} = \sum_{r=1}^{\infty} r^{-2}(1+k/r)^{-2}\{a_r \cos rx + b_r \sin rx\}$$

is the integrated series of a Fourier series, since the squares of the coefficients of $\cos rx$ and $\sin rx$ in its derived series form a convergent series. Thus T_1, and similarly T_2, T_3, and T_4, converge to integrals, whence also $G(x)$ is an integral, as was stated.

The corollary of § 10 therefore holds.

§ 13. We now require certain lemmas, of which the proof can be easily supplied by the reader. These replace in the theory of non-harmonic series the well known normal properties possessed by $\sin nx$ and $\cos nx$, where n is an integral. They are accordingly fundamental in the development of the theory of non-harmonic trigonometrical series, to which reference was made in the introduction.

LEMMA 1.—*If r and n are different integers, positive, negative, or zero,*

$$\int_0^{2\pi} \sin(n+k)(u-2\pi)\cos(r+k)u\,du = 0, \qquad (0 \leq k < 1)$$

and if $r = n$, the value of the integral is

$$-\pi \sin 2k\pi.$$

LEMMA 2.—*If r and n are integers, positive, negative, or zero, such that $r+n$ is not zero,*

$$\int_0^{2\pi} \sin(n-k)(u-2\pi)\cos(r+k)u\,du = 0$$

and if $r+n = 0$, the value of the integral is

$$\pi \sin 2k\pi.$$

§ 14. From these Lemmas we have at once, by subtraction and addition, the following formulæ :—

$$\left.\begin{aligned} \int_0^{2\pi} \sin k(u-2\pi)\cos nu\cos(r+k)u\,du &= 0, \quad (r \neq \pm n) \\ &= -\tfrac{1}{2}\pi \sin 2k\pi \quad (r = \pm n \neq 0) \\ &= -\pi \sin 2k\pi \quad (r = n = 0) \end{aligned}\right\}, \quad \text{(II)}$$

$$\left.\begin{aligned} \int_0^{2\pi} \cos k(u-2\pi)\sin nu\cos(r+k)u\,du &= 0, \quad (r \neq \pm n), \\ &= \tfrac{1}{2}\pi \sin 2k\pi, \quad (r = \pm n \neq 0) \\ &= 0, \quad (r = n = 0) \end{aligned}\right\}. \quad \text{(III)}$$

§ 15. Now consider the series

$$\sum_{r=-\infty}^{r=+\infty} \{a_r \cos(r+k)x + b_r \sin(r+k)x\} = \sum_{r=-\infty}^{r=+\infty} A_r^{(k)}, \qquad (0 < k < 1). \quad (1)$$

As before we suppose that the second integrated series

$$-\sum_{r=-\infty}^{r=+\infty} (r+k)^{-2} A_r^{(k)}$$

converges uniformly, and denote its sum-function by $G(x)$. In particular this will be the case if a_r and b_r are bounded functions of r.

We have then

$$\tfrac{1}{2}\{G(x+u)+G(x-u)\} = -(r+k)^{-2} \sum_{r=-\infty}^{r=+\infty} (r+k)^{-2} A_r^{(k)} \cos(r+k)u, \quad (2)$$

the series converging uniformly. We may, therefore, multiply both sides by $\sin k(u-2\pi)\cos ru$, or by $\cos k(u-2\pi)\sin ru$, and integrate term-by-term. Using the formula II, we then get

$$\int_0^{2\pi} \tfrac{1}{2}\{G(x+u)+G(x-u)\} \sin k(u-2\pi)\cos ru\,du = (r+k)^{-2} A_r^{(k)} \tfrac{1}{2}\pi \sin 2k\pi$$

and therefore, for $r \geq 0$.

$$A_r^{(k)} + A_{-r}^{(k)}$$
$$= -\frac{2}{\pi \sin 2k\pi}\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\sin k(u-2\pi)\cos ru\,du. \quad (3)$$

Similarly, using III, we have for $r \geq 1$,

$$A_r^{(k)} + A_{-r}^{(k)}$$
$$= -\frac{2}{\pi \sin 2k\pi}\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\cos k(u-2\pi)\sin ru\,du. \quad (3a)$$

Hence, since

$$\tfrac{1}{2}+\cos u+\cos 2u+\ldots+\cos nu = \tfrac{1}{2}\sum_{r=-n}^{r=+n}\cos ru = \tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}u\sin(n+\tfrac{1}{2})u$$
$$= \tfrac{1}{2}\sin nt\cot\tfrac{1}{2}u+\tfrac{1}{2}\cos nu,$$

we get for the partial summation $S_n^{(k)}$ of the series (1),

$$S_n^{(k)} = \sum_{r=-n}^{r=+n} A_n^{(k)} = -\frac{1}{\pi\sin 2k\pi}$$
$$\times\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\sin k(u-2\pi)\operatorname{cosec}\tfrac{1}{2}u\sin(n+\tfrac{1}{2})u\,du$$
$$= A_n^{(k)} - \frac{1}{\pi\sin 2k\pi}$$
$$\times\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\sin k(u-2\pi)\cot\tfrac{1}{2}u\sin nu\,du. \quad (4)$$

§ 16. The proof of the fundamental result stated in the introduction now at once follows.

We see by § 15 that the upper and lower functions of our series are, omitting an irrelevant factor, equal to the upper and lower limits, when $r\to\infty$, of the expression*

$$I \equiv \frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\sin k(u-2\pi)\cot\tfrac{1}{2}u\sin nu\,du.$$

In treating this expression the first step is to introduce, in place of $\sin k(u-2\pi)\cot\frac{1}{2}u$, a function $\phi(u)$, defined as equal to

$$\sin k(u-2\pi)\cot\tfrac{1}{2}u, \qquad (e < u \leq 2\pi),$$

and having at the origin the value $2k$, while its differential coefficient is zero there. In $(0, e)$ this function $\phi(u)$ is to be the integral of an integral, with the same left and right hand differential coefficients at the point e, and is otherwise subject to no conditions.

* We work here with $\sum_{n=-\infty}^{n=+\infty}$, whereas in the introduction it is $\sum_{n=1}^{\infty}$. This does not affect the result.

The function $\phi(u)-2k$ then satisfies the conditions imposed on $\lambda(u)$ in the extension of Riemann's Third Theorem (§ 10).
We thus get

$$\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}(\phi(u)-2k)\sin nu\,du \to 0, \qquad (n\to\infty),$$

Hence, in our investigation, we may replace I by

$$\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\{\sin k(u-2\pi)\cot\tfrac{1}{2}u-\phi(u)+2k\}\sin nu\,du$$

$$= 2k\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\sin nu\,du$$

$$+\frac{d^2}{dx^2}\int_0^{e}\{G(x+u)+G(x-u)\}\sin k(u-2\pi)\cot\tfrac{1}{2}u\sin nu\,du$$

$$-\frac{d^2}{dx^2}\int_0^{e}\{G(x+u)+G(x-u)\}\phi(u)\sin nu\,du.$$

Now if $G(x)$ is a repeated integral in $(x-e, x+e)$, we may in the second and third terms perform the repeated differentiation under the integral sign. Since $\phi(u)$ is bounded, the last term then tends to zero by the Theorem of Riemann-Lebesgue. The second term will then require further consideration, it is *the* important term, as it leads to the identification of our theory with the Fourier theory.

The first term we treat as follows: we introduce a new function $\psi(u)$, defined as equal to

$$1-\cos k(u-2\pi), \qquad (e\le u\le 2\pi),$$

having at the origin zero for its value, and for the value of its differential coefficient. This function, when suitably filled in from 0 to e, satisfies the conditions imposed on $\lambda(u)$ in the extended form of Riemann's Third Theorem. Hence, omitting the factor $2k$, our first term may be replaced by the following:—

$$\frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\{1-\psi(u)\}\sin nu\,du$$

$$= \frac{d^2}{dx^2}\int_0^{2\pi}\{G(x+u)+G(x-u)\}\cos k(u-2\pi)\sin nu\,du$$

$$-\frac{d^2}{dx^2}\int_0^{e}\{G(x+u)+G(x-u)\}\cos k(u-2\pi)\sin nu\,du$$

$$+\frac{d^2}{dx^2}\int_0^{e}\{G(x+u)+G(x-u)\}\sin nu\,du$$

$$-\frac{d^2}{dx^2}\int_0^{e}\{G(x+u)+G(x-u)\}\psi(u)\,du.$$

The last three terms tend to zero by the Theorem of Riemann-Lebesgue, since repeated differentiation under the sign of integration in $(0, e)$ is allowable, and the functions $\psi(u)$, $\cos k(u-2\pi)$ and unity are all bounded. The first term by $(3a)$ of § 15 is equal to

$$-\tfrac{1}{2}\pi \sin 2k\pi \, \mathrm{A}_n^{(k)},$$

and, therefore, by hypothesis, tends to zero.

Re-inserting the factor hitherto neglected we are thus, in the case when G is the integral of an integral in $(x-e, x+e)$, left with the expression

$$\frac{1}{\pi}\int_0^e \{f(x+u)+f(x-u)\}\frac{\sin k(2\pi-u)}{\sin 2k\pi}\cot\tfrac{1}{2}u \sin nu\, du.$$

Now

$$\left\{\frac{\sin k(2\pi-u)}{\sin 2k\pi}-1\right\}\cot\tfrac{1}{2}u$$

$$=\frac{\sin k(2\pi-u)-\sin 2k\pi}{\sin 2k\pi \tan\tfrac{1}{2}u} \longrightarrow \frac{-k\cos 2k\pi}{\tfrac{1}{2}}, \qquad (u\to 0),$$

and is therefore bounded in the neighbourhood of the origin. By the Theorem of Riemann-Lebesgue, therefore, multiplying by

$$\{f(x+u)+f(x-u)\}\sin nu$$

and integrating from 0 to e, the integral tends to zero as $n\to\infty$. Thus we may replace our expression by the Fourier expression

$$\frac{1}{\pi}\int_0^e \{f(x+u)+f(x-u)\}\cot\tfrac{1}{2}u \sin nu\, du.$$

This proves the theorem in this case.

In the general case the terms we are left with depend only on the values of G and ϕ in $(x-e, x+e)$, which proves the theorem.

Cor.—*If* $a_n\to 0$, $b_n\to 0$, *writing* $\mathrm{F}=\dfrac{d\mathrm{G}}{dx}$, *the upper and lower functions of* $\sum_{n=-\infty}^{n=+\infty} \mathrm{A}_n^{(k)}$ *are the upper and lower limits of*

$$\pi \sin 2k\pi \frac{d}{dx}\int_0^e \{\mathrm{F}(x+u)+\mathrm{F}(x-u)\}\{\sin k(2\pi-u)\cot\tfrac{1}{2}u-\phi(u)\}\sin nu\, du,$$

where ϕ *is the integral of an integral in* $(0, e)$ *and has the value* $2k$ *at the origin and* $\sin k(2\pi e)\cot\frac{1}{2}$ *at* e, *while its differential coefficient is zero at the origin and* $-k\cos k(2\pi-e)$ *at* e, *and is otherwise arbitrary.*